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Wigner functions for time-dependent coupled linear oscillators via linear and quadratic invariant processes

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Abstract
At exact resonance we derive the linear and quadratic invariants for time-dependent coupled linear oscillators in the presence of second harmonic generation. Employing these invariants we introduce an accurate definition of the Dirac operators from which the wavefunctions in both coherent and Fock (number) states representations are calculated. We also derive the W-Wigner function of an arbitrary state at time t, evolving in the system under consideration. Moreover we calculate the correlation coefficient between position and momentum and we find that it is identical in both fields whatever the state used.

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1. Introduction

In the last few decades the problem of time-dependent linear oscillators (variable mass and/or variable frequency) has played a major role in the study of several phenomena of physics [1–10]. A great deal of attention has been paid to some specific problems of time-dependent oscillators: the damped linear oscillator and the strongly pulsating oscillator, for which the mass is taken to be a function of time. For example in quantum optics one can see the quantum treatment of a decaying oscillator that originated from an interest in a cavity oscillator in which the electromagnetic field varies with time under the action of some reservoir as, for instance, in laser production [11]. In fact these two specific problems have been handled extensively in different directions by many authors by whom closed-form solutions of the wavefunction in the Schrödinger picture as well as the equations of motion in the Heisenberg picture are obtained in explicit and compact form, see for example [5–9]. Here we may also refer to the problem of a sudden change in mass during which the observation of the squeezing phenomenon is reported [12]. It is interesting to note that the existence of the fluctuations in the case of time-dependent mass is responsible for the second harmonic generation and consequently one may expect to observe the squeezing phenomenon in one of the components of the quadrature.

In the meantime the problem of weakly pulsating as well as damped pulsating oscillators has also been considered, for which the dynamical operators and the wavefunction in the Schrödinger picture have been obtained, however, under restricted conditions [13]. This indicates that the problem of the time-dependent linear oscillator (variable mass or variable frequency) is not an easy task to handle. However, one may avoid the direct consideration of the problem and deal instead with the constants of the motion (invariants) [14–21]. In this case the appearance of the well-known Ermakov–Pinney equation [22, 23, 27] which is a nonlinear differential equation with variable coefficients should be expected. On the other hand the construction of the invariant operators (constants of the motion) in quantum mechanics has attracted much attention, for example the authors of [14] introduced in their paper the role of these operators, which describe a quantum system governed by a time-dependent Hamiltonian. They have shown that, if the system admits an invariant $\hat{I}(t)$ among its observables, it is possible to find a privileged basis of eigenstates of this operator for which the expansion of the state vector on this basis can be performed with time-independent coefficients. Recently Schrade *et al* have used this concept and employed the time-dependent linear oscillator to construct an operator constant of the motion in order to discuss the Wigner function in the Paul trap [24, 25]. Their discussions included the calculation of the correlation coefficient between position and momentum which appears in the Schrödinger uncertainty relation. The Hamiltonian model adopted in that work is given by

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}\omega_0^2(t)m\hat{q}^2, \tag{1.1}$$

where the frequency of the oscillator is considered to be time-dependent rather than constant in order to discuss the Paul trap. In this paper we seek to handle a problem different to that of reference [24, 25]. This problem is the interaction between two coupled oscillators under the influence of second harmonic generation (degenerate parametric amplifier). We assume that the coupling parameters between the fields as well as the self-coupling response of the second harmonic generation are time-dependent. In this case the Hamiltonian governing such a system takes the form

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{2}\left[\hat{p}_i^2/m_i + \omega_i^2(t)m_i\hat{q}_i^2 + \gamma_i(t)(\hat{q}_i\hat{p}_i + \hat{p}_i\hat{q}_i)\right] - \lambda(t)\hat{q}_1\hat{q}_2, \tag{1.2}$$

where $\omega_i(t)$ is the field frequency, m_i is the mass, $\gamma_i(t)$ and $\lambda(t)$ are the parametric responses of the second harmonic generation and the field coupling parameter, respectively. The above Hamiltonian can be regarded as a generalization of the Hamiltonian model given in [24]. The main purpose of this work is to find the time evolution of an arbitrary Wigner function and to examine the effect of the time upon its behaviour. This can be achieved if one manages to derive the explicit expression of the dynamical operators $\hat{q}_i$ and $\hat{p}_i$, $i = 1, 2$, or if we manage to obtain the explicit solution of the wavefunction in the Schrödinger picture. This cannot be done for any arbitrary functions of the time. However, if we employ the idea of the constants of the motion, then we overcome any difficulty. For this reason we devote the following section to considering the constants of the motion for the present system. This is followed by the construction of the wavefunction in both the number states (Schrödinger picture) as well as in the coherent states in section 3. In section 4 we calculate the Wigner function and in section 5 the phase space distribution function. The correlation coefficient is obtained in section 6 and in section 7 we present our conclusions.

2. Linear and quadratic invariants

We devote this section to determining the constants of the motion for the present system. It is more advantageous to concentrate on finding the linear and quadratic invariants for the Hamiltonian (1.2). First we consider the fundamental first-degree invariants.

2.1. Linear invariant

To determine the fundamental linear invariants we introduce a constant of the motion $\hat{I}^{(1)}(t)$ such that

$$\hat{I}^{(1)}(t) = \sum_{i=1}^{2} (\zeta_i(t)\hat{p}_i + \beta_i(t)\hat{q}_i), \qquad i = 1, 2, \tag{2.1}$$

where $\zeta_i(t)$ and $\beta_i(t)$ are as yet unspecified functions of time. To establish that the operator $\hat{I}^{(1)}(t)$ is a constant of the motion, one needs to find explicit expressions for the time-dependent complex functions $\zeta_i(t)$ and $\beta_i(t)$. This can be achieved if we use the equation

$$\frac{\mathrm{d}\hat{I}^{(1)}}{\mathrm{d}t} = \frac{\partial \hat{I}^{(1)}}{\partial t} + \sum_{i=1}^{2} \left\{ \frac{\partial \hat{I}^{(1)}}{\partial \hat{q}_i} \frac{\partial \hat{H}}{\partial \hat{p}_i} - \frac{\partial \hat{I}^{(1)}}{\partial \hat{p}_i} \frac{\partial \hat{H}}{\partial \hat{q}_i} \right\} = 0. \tag{2.2}$$

In this case, if one uses equation (1.2) together with equations (2.1) and (2.2), we have

$$\frac{\mathrm{d}\zeta_i}{\mathrm{d}t} + \beta_i = \gamma_i \zeta_i, \qquad \frac{\mathrm{d}\beta_i}{\mathrm{d}t} + \gamma_i \beta_i = \omega_i^2 \zeta_i - \lambda \zeta_j, \qquad i \neq j = 1, 2, \tag{2.3}$$

and after eliminating $\beta_i(t)$ we obtain

$$\frac{\mathrm{d}^2\zeta_1}{\mathrm{d}t^2} + \Omega_1^2(t)\zeta_1 = \lambda(t)\zeta_2, \qquad \frac{\mathrm{d}^2\zeta_2}{\mathrm{d}t^2} + \Omega_2^2(t)\zeta_2 = \lambda(t)\zeta_1, \tag{2.4}$$

where $\Omega_i^2(t) = \left[\omega_i^2(t) - \gamma_i^2(t) - \dot{\gamma}_i(t)\right]$, $i = 1, 2$. Now, if one manages to obtain the explicit form of $\zeta_i(t)$, then it is easy to find the corresponding expression for the $\beta_i(t)$. There are two classes of first-degree invariants which can be obtained. Thus we have

$$\hat{I}_q^{(1)}(t) = \sum_{i=1}^{2} [\zeta_i(t)\hat{p}_i + (\gamma_i\zeta_i - \dot{\zeta}_i)\hat{q}_i] \tag{2.5}$$

and

$$\begin{aligned} \hat{I}_p^{(1)}(t) = \sum_{i=1}^{2} \beta_i(t)\hat{q}_i + \{ &\left[\omega_2^2(\dot{\beta}_1(t) + \gamma_1\beta_1(t)) + \lambda(\dot{\beta}_2(t) + \gamma_2\beta_2(t))\right]\hat{p}_1 \\ &+ \left[\omega_1^2(\dot{\beta}_2(t) + \gamma_2\beta_2(t)) + \lambda(\dot{\beta}_1(t) + \gamma_1\beta_1(t))\right]\hat{p}_2 \} / \left(\omega_1^2\omega_2^2 - \lambda^2\right), \end{aligned} \tag{2.6}$$

where overdot denotes differentiation with respect to the time. At the exact resonance, i.e., when $\omega_1 = \omega_2 = \omega$, we have

$$\begin{aligned} \hat{I}_p^{(1)}(t) = \sum_{i=1}^{2} &\left\{ \beta_i(t)\hat{q}_i + \frac{\omega^2(t)}{[\omega^4(t) - \lambda^2(t)]}[\dot{\beta}_i(t) + \gamma_i\beta_i(t)]\hat{p}_i \right\} \\ &+ \frac{\lambda(t)}{[\omega^4(t) - \lambda^2(t)]} \sum_{i\neq j=1}^{2} (\dot{\beta}_i(t) + \gamma_i\beta_i(t))\hat{p}_j. \end{aligned} \tag{2.7}$$

Since we are dealing with a system of two coupled oscillators, to reach our goal we have to construct from the constants of the motion two pairs of creation and annihilation operators playing roles similar to those of the Dirac operators. For this purpose we define the functions $\zeta_i(t)$ as follows

$$\zeta_1(t) = k_+(t) + k_-(t) \qquad \text{and} \qquad \zeta_2(t) = k_+(t) - k_-(t). \tag{2.8}$$

In this case the new functions $k_\pm(t)$ satisfy the equations

$$\frac{\mathrm{d}^2 k_+}{\mathrm{d}t^2} + \Omega_+^2(t)k_+ = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 k_-}{\mathrm{d}t^2} + \Omega_-^2(t)k_- = 0, \tag{2.9}$$

where $\Omega_\pm^2(t) = \{\Omega^2(t) \mp \lambda(t)\}$ and we have assumed $\Omega_1^2(t) = \Omega_2^2(t) = \Omega^2(t)$ (the exact resonance case). Thus, if one uses equation (2.5), one can construct Dirac operators in the form

$$\begin{aligned} \hat{A}_1(t) &= \frac{1}{2\sqrt{\hbar}}[(\gamma_1 k_+ - \dot{k}_+)\hat{q}_1 + (\gamma_2 k_+ - \dot{k}_+)\hat{q}_2 + k_+(\hat{p}_1 + \hat{p}_2)], \\ \hat{A}_2(t) &= \frac{1}{2\sqrt{\hbar}}[(\gamma_2 k_- - \dot{k}_-)\hat{q}_2 - (\gamma_1 k_- - \dot{k}_-)\hat{q}_1 + k_-(\hat{p}_2 - \hat{p}_1)]. \end{aligned} \tag{2.10}$$

It is easy to check that the above two operators satisfy the commutation relation $\left[\hat{A}_i(t), \hat{A}_j^\dagger(t)\right] = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, provided we take the Wronskian $\mathcal{W}$ as

$$\mathcal{W} = [k_\pm^*(t)\dot{k}_\pm(t) - \dot{k}_\pm^*(t)k_\pm(t)] = 2\mathrm{i}, \tag{2.11}$$

in which the subscripts $\pm$ vary independently in the first and second members of each term and $k_\pm(0) = 1$, and $\dot{k}_\pm(0) = \mathrm{i}$. It should be noted that in our calculations and for simplicity we have taken the mass to be unity. Now we turn our attention to consider the quadratic invariants for the present system.

2.2. *Quadratic invariants*

To continue our progress we introduce a second-degree invariant $\hat{I}^{(2)}(t)$ of the form

$$\hat{I}^{(2)}(t) = \sum_{i=1}^{2}\left[\overline{\zeta}_i(t)\hat{q}_i^2 + \overline{\beta}_i(t)\hat{p}_i^2 + 2\overline{\gamma}_i(t)\hat{q}_i\hat{p}_i\right] + (\mu_1\hat{q}_1\hat{q}_2 + \mu_2\hat{p}_1\hat{p}_2 + \mu_3\hat{q}_1\hat{p}_2 + \mu_4\hat{p}_1\hat{q}_2) \tag{2.12}$$

which together with equations (1.2) and (2.2) would lead to the complicated situation in which we would have to solve ten simultaneous differential equations. To avoid this complication it is more convenient for us to diagonalize the Hamiltonian (1.2). This may be achieved under the canonical transformation

$$\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\\ \hat{q}_1\\ \hat{q}_2\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi & 0 & 0\\ -\sin\phi & \cos\phi & 0 & 0\\ 0 & 0 & \cos\phi & \sin\phi\\ 0 & 0 & -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\hat{P}_1\\ \hat{P}_2\\ \hat{Q}_1\\ \hat{Q}_2\end{pmatrix}, \tag{2.13}$$

where $[\hat{q}_i, \hat{p}_j] = [\hat{Q}_i, \hat{P}_j] = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. In this case the Hamiltonian (1.2) takes the form

$$\hat{H}(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\hat{P}_i^2 + \overline{\Omega}_i^2(t)\hat{Q}_i^2 + \gamma(t)(\hat{Q}_i\hat{P}_i + \hat{P}_i\hat{Q}_i)\right] \tag{2.14}$$

provided we take $\phi = \pi/4$ and both of the response functions γ_i, $i = 1, 2$, are equal. The augmented frequencies $\overline{\Omega}_i$, $i = 1, 2$, are given by

$$\overline{\Omega}_1(t) = \sqrt{\omega^2(t) + \lambda(t)}, \qquad \overline{\Omega}_2(t) = \sqrt{\omega^2(t) - \lambda(t)}. \tag{2.15}$$

It is easy to realize that the Hamiltonian (2.14) is now separable and therefore we may construct the quadratic invariant in the form

$$\hat{I}^{(2)}(t) = \sum_{i=1}^{2} \left[\overline{\zeta}_i(t)\hat{Q}_i^2 + \overline{\beta}_i(t)\hat{P}_i^2 + 2\overline{\mu}_i(t)\hat{Q}_i\hat{P}_i \right], \tag{2.16}$$

where $\overline{\zeta}_i(t)$, $\overline{\beta}_i(t)$ and $\overline{\mu}_i(t)$ are real functions of the time. This in fact leads to the system of differential equations given by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\zeta}_i) + 2\gamma(t)\overline{\zeta}_i = 2\overline{\Omega}_i^2(t)\overline{\mu}_i, \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\beta}_i) - 2\gamma(t)\overline{\beta}_i = -2\overline{\mu}_i, \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\mu}_i) + \overline{\zeta}_i = \overline{\Omega}_i^2(t)\overline{\beta}_i.
\end{aligned} \tag{2.17}$$

Now, if we take $\overline{\beta}_i(t) = k_i^{\frac{1}{2}}\sigma_i^2(t)$ where k_i, $i = 1, 2$, are some constants, straightforward calculation leads us to express our results in terms of auxiliary functions $\sigma_i(t)$ that satisfy the Ermakov–Pinney equation [22, 27]

$$\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}t^2} + \Omega_-^2(t)\sigma_1 = \frac{1}{\sigma_1^3}, \qquad \frac{\mathrm{d}^2\sigma_2}{\mathrm{d}t^2} + \Omega_+^2(t)\sigma_2 = \frac{1}{\sigma_2^3}, \tag{2.18}$$

where the $\Omega_\pm(t)$ are given by equation (2.9).

The solution of such type of equation can be written in the form [27]

$$\sigma(t) = \left(ax_1^2 + bx_2^2 + 2cx_1x_2\right)^{\frac{1}{2}}, \tag{2.19}$$

where x_1 and x_2 are two linearly independent solutions of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \Omega_\pm^2 x = 0 \tag{2.20}$$

and the constants a, b and c are related according to

$$w^{-2} = (ab - c^2), \qquad w = x_1^2\frac{\mathrm{d}}{\mathrm{d}t}(x_2/x_1). \tag{2.21}$$

In this case we can construct the invariant

$$\hat{I}_Q^{(2)}(t) = \sum_{i=1}^{2} k_i^{\frac{1}{2}}\left[\hat{Q}_i^2/\sigma_i^2(t) + \{\sigma_i\hat{P}_i + (\gamma\sigma_i - \dot{\sigma}_i)\hat{Q}_i\}^2\right]. \tag{2.22}$$

Similarly, if we set $\overline{\zeta}_i = k_i^{\frac{1}{2}}\rho_i^2(t)$, then we can construct another form for the second-degree invariant as

$$\hat{I}_P^{(2)}(t) = \sum_{i=1}^{2} k_i^{\frac{1}{2}}\left[\hat{P}_i^2/\rho_i^2(t) + \left\{\rho_i\hat{Q}_i + \left(\frac{\dot{\rho}_i + \gamma\rho_i}{\Omega_i^2(t)}\right)\hat{P}_i\right\}^2\right], \tag{2.23}$$

where the auxiliary functions $\rho_i(t)$ satisfy the equation

$$\frac{\mathrm{d}^2\rho_i}{\mathrm{d}t^2} - 2\frac{\dot{\Omega}_i}{\Omega_i}\frac{\mathrm{d}\rho_i}{\mathrm{d}t} + \left(\Omega_i^2(t) - \gamma^2 + \dot{\gamma} - 2\gamma\frac{\dot{\Omega}_i}{\Omega_i}\right)\rho_i(t) = \frac{\Omega_i^4(t)}{\rho_i^3(t)}, \qquad i = 1, 2. \tag{2.24}$$

Now, if one uses equation (2.22), then it is possible to construct two pairs of Dirac operators in the form

$$\begin{aligned}
\overline{\hat{A}}_i(t) &= \frac{1}{\sqrt{2\hbar}}\left[\hat{Q}_i\left\{\frac{1}{\sigma_i} + \mathrm{i}(\gamma\sigma_i - \dot{\sigma}_i)\right\} + \mathrm{i}\sigma_i\hat{P}_i\right], \\
\overline{\hat{A}}_i^\dagger(t) &= \frac{1}{\sqrt{2\hbar}}\left[\hat{Q}_i\left\{\frac{1}{\sigma_i} - \mathrm{i}(\gamma\sigma_i - \dot{\sigma}_i)\right\} - \mathrm{i}\sigma_i\hat{P}_i\right]
\end{aligned} \tag{2.25}$$

which obey the usual commutation relation $\left[\overline{\hat{A}}_i, \overline{\hat{A}}_j^{\dagger}\right] = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. As the second task of this work is to find the wavefunction, this may be done if one uses either equation (2.25) or (2.10). However, we use both equations in this paper. This is considered in the following section.

3. The wavefunctions

In this section we turn our attention to find the wavefunction in both number states (Schrödinger picture) and coherent states. To reach this goal we use the annihilation and creation operators, $\hat{A}_i(t)$ and $\hat{A}_i^{\dagger}(t)$, given by equation (2.10) together with the coherent states

$$|\boldsymbol{\alpha}; t\rangle = \exp\left[-\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)\right] \sum_{n_1,n_2=0}^{\infty} \frac{\alpha_1^{n_1}\alpha_2^{n_2}}{\sqrt{n_1!n_2!}} |\boldsymbol{n}; t\rangle, \tag{3.1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$ and $\boldsymbol{n} = (n_1, n_2)^T$. The eigenvalues of the operators $\hat{A}_i(t)$ with respect to the coherent states are given by

$$\hat{A}_i(t)|\boldsymbol{\alpha}; t\rangle = \alpha_i(t)|\boldsymbol{\alpha}; t\rangle, \qquad \mathrm{i} = 1, 2.$$

For the present case we have

$$\psi_{\alpha}(\boldsymbol{q}, t) = N_{\alpha}(t) \exp\left[-\frac{\mathrm{i}}{2\hbar}\left\{\left[\gamma - \frac{\mathrm{d}}{\mathrm{d}t}(\log\sqrt{k_-k_+})\right]\left(q_1^2 + q_2^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\log(k_-/k_+)\right) q_1q_2\right\}\right]$$
$$\times \exp\left[\frac{\mathrm{i}}{\sqrt{\hbar}k_+k_-}\left\{(\alpha_1k_- - \alpha_2k_+)\, q_1 + (\alpha_1k_- + \alpha_2k_+)\, q_2\right\}\right], \tag{3.2}$$

where $\boldsymbol{q} = (q_1, q_2)^T$ and $N_{\alpha}(t)$ is the normalization constant which in general depends upon time and can be obtained from the relation

$$\int_{-\infty}^{\infty} |\psi_{\alpha}(\boldsymbol{q}, t)|^2\, \mathrm{d}\boldsymbol{q} = 1. \tag{3.3}$$

After minor algebra we find that the normalization constant $N_{\alpha}(t)$ can be written in the form

$$N_{\alpha}(t) = \frac{1}{(\pi\hbar)^{\frac{1}{2}}} |k_+k_-|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}[|\alpha_1|^2 + |\alpha_2|^2] + \frac{1}{2}\left(\alpha_1^2k_+^*/k_+ + \alpha_2^2k_-^*/k_-\right)\right). \tag{3.4}$$

Now, if one substitutes equation (3.4) into equation (3.2), we can obtain the wavefunction in the number (Fock) states representation. Thus

$$\psi_{\boldsymbol{n}}(\boldsymbol{q}, t) = \frac{|k_+k_-|^{-\frac{1}{2}}}{(\pi\hbar)^{\frac{1}{2}}} 2^{-(n_1+n_2)/2} \frac{(k_+^*/k_+)^{n_1/2}(k_-^*/k_-)^{n_2/2}}{\sqrt{n_1!n_2!}} H_{n_1}\left(\frac{(q_1 + q_2)}{|k_+|\sqrt{2\hbar}}\right) H_{n_2}\left(\frac{(q_2 - q_1)}{|k_-|\sqrt{2\hbar}}\right)$$
$$\times \exp\left[-\frac{\mathrm{i}}{2\hbar}\left\{\left[\gamma - \frac{\mathrm{d}}{\mathrm{d}t}\left(\log\sqrt{k_-k_+}\right)\right]\left(q_1^2 + q_2^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\log(k_-/k_+)\right) q_1q_2\right\}\right], \tag{3.5}$$

where $H_n(.)$ is the Hermite polynomial of order n. Since $k_{\pm}(t)$ are the solutions of the *ordinary differential equations* given by (2.9), our result in this case is too limited. However, if we employ the Dirac operators (obtained from the second-degree invariants) for finding the wavefunction in the Schrödinger picture, then we are able in this case to have more flexibility in handling the present problem. This in fact is due to the dependence of the wavefunction in the second case on the solution of *nonlinear differential equations* (the Ermakov–Pinney equation, (2.18) or (2.20)). Thus, after some calculation, we obtain the wavefunction in the coherent state representation in the form

$$\begin{aligned}\Psi_\alpha(\boldsymbol{q},t) = {}& \frac{(\pi\hbar)^{-\frac{1}{2}}}{\sqrt{\sigma_1\sigma_2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{2}\left(|\alpha_i|^2+\alpha_i^2\right)\right] \\ & \times \exp\left\{\frac{1}{\sqrt{\hbar}}\left[\left(\frac{\alpha_1}{\sigma_1}+\frac{\alpha_2}{\sigma_2}\right)q_1+\left(\frac{\alpha_2}{\sigma_2}-\frac{\alpha_1}{\sigma_1}\right)q_2\right]\right\} \\ & \times \exp\left\{-\frac{1}{4\hbar}\left[\left(\sigma_1^{-2}+\sigma_2^{-2}\right)+\mathrm{i}\left(2\gamma-\frac{\mathrm{d}}{\mathrm{d}t}\left(\log(\sigma_1\sigma_2)\right)\right)\right]\left(q_1^2+q_2^2\right)\right\} \\ & \times \exp\left\{-\frac{1}{2\hbar}\left[\left(\sigma_2^{-2}-\sigma_1^{-2}\right)+\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\log(\sigma_1/\sigma_2)\right)\right]q_1q_2\right\}\end{aligned} \tag{3.6}$$

and for the number (Fock) state representation we have

$$\begin{aligned}\Psi_n(\boldsymbol{q},t) = {}& \frac{(\pi\hbar)^{-\frac{1}{2}}}{\sqrt{\sigma_1\sigma_2}} 2^{-(n_1+n_2)/2}\frac{1}{\sqrt{n_1!n_2!}}H_{n_1}\left(\frac{(q_1-q_2)}{\sigma_1\sqrt{2\hbar}}\right)H_{n_2}\left(\frac{(q_2+q_1)}{\sigma_2\sqrt{2\hbar}}\right) \\ & \times \exp\left\{-\frac{1}{4\hbar}\left[\left(\sigma_1^{-2}+\sigma_2^{-2}\right)+\mathrm{i}\left(2\gamma-\frac{\mathrm{d}}{\mathrm{d}t}\left(\log(\sigma_1\sigma_2)\right)\right)\right]\left(q_1^2+q_2^2\right)\right\} \\ & \times \exp\left\{-\frac{1}{2\hbar}\left[\left(\sigma_2^{-2}-\sigma_1^{-2}\right)+\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\log(\sigma_1/\sigma_2)\right)\right]q_1q_2\right\}.\end{aligned} \tag{3.7}$$

Having obtained the wavefunction in both the coherent and number state representations we are in a position to calculate the W-Wigner function with respect to the coherent and number states. This is treated in the following section.

4. Wigner functions

In the following we calculate the W-Wigner function by employing the wavefunctions for both number state and coherent state representations obtained in the previous section. The Wigner function can be calculated for an arbitrary state if one uses the relation

$$W(\boldsymbol{q},\boldsymbol{p},t) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}y_1\,\mathrm{d}y_2\prod_{j=1}^{2}\Psi(q_j-y_j,t)\Psi^*(q_j+y_j,t)\exp\left[\frac{2\mathrm{i}}{\hbar}\sum_{j=1}^{2}p_jy_j\right]. \tag{4.1}$$

After lengthy but straightforward calculations we find that the W-Wigner function for the coherent state representation takes the form

$$\begin{aligned}W_\alpha(\boldsymbol{q},\boldsymbol{p},t) = {}& \frac{1}{\pi^2}\exp\left\{-\frac{1}{2\hbar}\left[(|f_+|^2+|f_-|^2)\left(q_1^2+q_2^2\right)+2(|f_+|^2-|f_-|^2)q_1q_2\right]\right\} \\ & \times \exp\left\{-\frac{1}{2\hbar}\left[(|k_+|^2+|k_-|^2)\left(p_1^2+p_2^2\right)+2(|k_+|^2-|k_-|^2)p_1p_2\right]\right\} \\ & \times \exp\left\{-\frac{1}{2\hbar}\left[(k_+f_+^*+k_+^*f_+)+(k_-f_-^*+k_-^*f_-)\right](q_1p_1+q_2p_2)\right\} \\ & \times \exp\left\{-\frac{1}{2\hbar}[(k_+f_+^*+k_+^*f_+)-(k_-f_-^*+k_-^*f_-)](p_1q_2+p_2q_1)\right\} \\ & \times \exp\left\{\frac{1}{\sqrt{\hbar}}[(\alpha_1f_+^*+\alpha_1^*f_+)+(\alpha_2f_-^*+\alpha_2^*f_-)]q_1\right\}\end{aligned}$$

$$\times \exp\left\{-\frac{1}{\sqrt{\hbar}}[(\alpha_2 f_-^* + \alpha_2^* f_-) - (\alpha_1 f_+^* + \alpha_1^* f_+)]q_2\right\}$$
$$\times \exp\left\{\frac{1}{\sqrt{\hbar}}[(\alpha_1 k_+^* + \alpha_1^* k_+) + (\alpha_2 k_-^* + \alpha_2^* k_-)]p_1\right\}$$
$$\times \exp\left\{-\frac{1}{\sqrt{\hbar}}[(\alpha_2 k_-^* + \alpha_2^* k_-) - (\alpha_1 k_+^* + \alpha_1^* k_+)]p_2\right\}$$
$$\times \exp\{-2[|\alpha_1|^2 + |\alpha_2|^2]\} \tag{4.2}$$

or the more compact form

$$W_\alpha(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{1}{\pi^2}\exp\left\{-\left|\left(\frac{1}{\sqrt{2\hbar}}\,[f_-(q_1 - q_2) + k_-(p_1 - p_2)] - \sqrt{2}\alpha_2\right)\right|^2\right\}$$
$$\times \exp\left\{-\left|\left(\frac{1}{\sqrt{2\hbar}}\,[f_+(q_2 + q_1) + k_+(p_2 + p_1)] - \sqrt{2}\alpha_1\right)\right|^2\right\}, \tag{4.3}$$

where we have defined

$$f_\pm^* = \left[k_\pm^*\left(\gamma - \frac{\mathrm{d}}{\mathrm{d}t}(\log|k_\pm|)\right) + \frac{\mathrm{i}}{k_\pm}\right], \qquad |f_\pm|^2 = \left[|k_\pm|'^2 + |k_\pm|^2\left(\gamma^2 - 2\gamma\frac{|k_\pm|'}{|k_\pm|}\right)\right] \tag{4.4}$$

and the prime denotes differentiation with respect to the time.

Similarly one can obtain the W-Wigner function for the number state representation if one uses the identity

$$\exp(ax + by + cxy) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(by)^m}{m!}\left(\frac{cx}{b}\right)^n L_n^{(m-n)}\left(-\frac{ab}{c}\right) \tag{4.5}$$

together with the coherent states given by equation (3.1). In the present case we find that

$$W_n(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{(-)^{n_1+n_2}}{\pi^2}\exp\left\{-\frac{1}{2\hbar}\left[(|f_+|^2 + |f_-|^2)\left(q_1^2 + q_2^2\right) + 2(|f_+|^2 - |f_-|^2)q_1q_2\right]\right\}$$
$$\times \exp\left\{-\frac{1}{2\hbar}\left[(|k_+|^2 + |k_-|^2)\left(p_1^2 + p_2^2\right) + 2(|k_+|^2 - |k_-|^2)p_1p_2\right]\right\}$$
$$\times \exp\left\{-\frac{1}{2\hbar}[(k_+f_+^* + k_+^*f_+) + (k_-f_-^* + k_-^*f_-)](q_1p_1 + q_2p_2)\right\}$$
$$\times \exp\left\{-\frac{1}{2\hbar}[(k_+f_+^* + k_+^*f_+) - (k_-f_-^* + k_-^*f_-)](p_1q_2 + p_2q_1)\right\}$$
$$\times L_{n_1}\left(\frac{1}{\hbar}|f_+[q_2 + q_1] + k_+[p_2 + p_1]|^2\right)L_{n_2}\left(\frac{1}{\hbar}|f_-[q_1 - q_2] + k_-[p_1 - p_2]|^2\right), \tag{4.6}$$

where $L_n(.)$ is the Laguerre polynomial of order n. Alternately we may obtain another expression for the Wigner function using equations (3.6) and (3.7). For the coherent state representation in this case we have

$$W_\alpha(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{1}{\pi^2}\exp\left[-\sum_{j=1}^{2}\left|\left(\frac{z_j(t)}{\sqrt{\hbar}} - \sqrt{2}\alpha_j\right)\right|^2\right], \tag{4.7}$$

where z_j is a complex quantity defined as

$$z_j(t) = \left[\sigma_j^{-1}(t) + \mathrm{i}(\gamma\sigma_j(t) - \sigma_j'(t))\right]Q_j + \mathrm{i}\sigma_j(t)P_j, \qquad j = 1, 2, \tag{4.8}$$

and the Q_j and P_j are given by equation (2.13), while σ_j is a solution of the Ermakov–Pinney equation (2.18). For the number state representation we have the expression

$$W_{\boldsymbol{n}}(\boldsymbol{q},\boldsymbol{p},t) = \frac{(-)^{n_1+n_2}}{\pi^2}\exp\left[-\frac{1}{\hbar}(|z_1(t)|^2+|z_2(t)|^2)\right]L_{n_1}\left(\frac{1}{\hbar}|z_1(t)|^2\right)L_{n_2}\left(\frac{1}{\hbar}|z_2(t)|^2\right). \tag{4.9}$$

It should be noted that to obtain the probability density, $|\widetilde{\Psi}(\boldsymbol{q},t)|^2$, of finding the position $\boldsymbol{q}$ one needs to integrate the W-Wigner function along a path, which is parallel to the momentum axis and which goes through $\boldsymbol{q}$, such that

$$|\widetilde{\Psi}(\boldsymbol{q},t)|^2 = \int_{-\infty}^{\infty} W(\boldsymbol{q},\boldsymbol{p},t)\,\mathrm{d}\boldsymbol{p}. \tag{4.10}$$

Similarly, to obtain the probability density $|\widetilde{\Psi}(\boldsymbol{p},t)|^2$ of finding the momentum $\boldsymbol{p}$ one needs to integrate the W-Wigner function along a path, which is parallel to the position axis and through $\boldsymbol{p}$, such that

$$|\widetilde{\Psi}(\boldsymbol{p},t)|^2 = \int_{-\infty}^{\infty} W(\boldsymbol{q},\boldsymbol{p},t)\,\mathrm{d}\boldsymbol{q}. \tag{4.11}$$

In the forthcoming section we employ the W-Wigner function to calculate both the phase distribution function $\mathcal{R}_{\theta,\phi}$ and the distribution function $\mathcal{R}_\alpha$.

5. Phase space

To calculate the phase space distribution function one may evaluate the integral

$$\mathcal{R}_{\theta_1,\theta_2}(\boldsymbol{q},\boldsymbol{p},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\alpha(\boldsymbol{q},\boldsymbol{p},t)|\alpha_1||\alpha_2|\,\mathrm{d}(|\alpha_1|)\,\mathrm{d}(|\alpha_2|). \tag{5.1}$$

As before we introduce two expressions. One is related to the linear invariants while the other is related to the two quadratic invariants. For the first case a straightforward calculation leads to

$$\begin{aligned}\mathcal{R}_{\theta_1,\theta_2}(\boldsymbol{q},\boldsymbol{p},t) &= \frac{1}{16\pi^2}\exp\left[-\frac{1}{\hbar}(|z_+|^2+|z_-|^2)\right]\left[1-2\beta_+\exp\left(\beta_+^2\right)\mathrm{erfc}(\beta_+)\right]\\ &\quad\times[1-2\beta_-\exp(\beta_-^2)\,\mathrm{erfc}(\beta_-)],\end{aligned} \tag{5.2}$$

where

$$z_+ = \tfrac{1}{\sqrt{2}}[f_+(q_2+q_1)+k_+(p_2+p_1)], \qquad z_- = \tfrac{1}{\sqrt{2}}[f_-(q_1-q_2)+k_-(p_1-p_2)] \tag{5.3}$$

and $\mathrm{erfc}(\beta_\pm)$ is the complement of the error function $\mathrm{erf}(\beta_\pm)$ given by

$$\mathrm{erf}\,\beta_\pm = \frac{4}{\pi}\sum_{k=0}^{\infty}(-)^k\frac{(\beta_\pm)^{2k+1}}{k!(2k+1)}, \qquad \beta_\pm = -\frac{z_\pm}{\sqrt{\hbar}}\cos\theta_\pm \tag{5.4}$$

and $\theta_\pm$ is the phase factor for the coherent states.

A similar expression is found from the quadratic invariants as

$$\begin{aligned}\mathcal{R}_{\theta_1,\theta_2}(\boldsymbol{q},\boldsymbol{p},t) &= \frac{1}{16\pi^2}\exp\left[-\frac{1}{\hbar}(|z_1|^2+|z_2|^2)\right]\left[1-2\beta_1\exp\left(\beta_1^2\right)\mathrm{erfc}(\beta_1)\right]\\ &\quad\times\left[1-2\beta_2\exp\left(\beta_2^2\right)\mathrm{erfc}(\beta_2)\right],\end{aligned} \tag{5.5}$$

where $z_i, i = 1, 2$, are given by equation (4.8) and $\beta_i, i = 1, 2$, have the same meanings as $\beta_\pm$.

On the other hand, we can evaluate the distribution function

$$\mathcal{R}_\alpha(\boldsymbol{q},\boldsymbol{p},t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W_\alpha(\boldsymbol{q},\boldsymbol{p},t)\,\mathrm{d}(\mathrm{Im}(\alpha_1))\,\mathrm{d}(\mathrm{Im}(\alpha_2)). \tag{5.6}$$

For the linear invariant case and after straightforward calculation we have

$$\mathcal{R}_{\mathrm{Re}(\alpha)}(\boldsymbol{q},\boldsymbol{p},t)=\frac{2}{\pi}\exp\left[-2\left(\left[u_1-\frac{1}{4}F_+(\boldsymbol{q},\boldsymbol{p},t)\right]^2+\left[u_2-\frac{1}{4}F_-(\boldsymbol{q},\boldsymbol{p},t)\right]^2\right)\right], \tag{5.7}$$

where

$$\begin{aligned}F_+(\boldsymbol{q},\boldsymbol{p},t)&=\frac{1}{\sqrt{\hbar}}[(f_++f_+^*)(q_1+q_2)+(k_++k_+^*)(p_1+p_2)],\\ F_-(\boldsymbol{q},\boldsymbol{p},t)&=\frac{1}{\sqrt{\hbar}}[(f_-+f_-^*)(q_1-q_2)+(k_-+k_-^*)(p_1-p_2)]\end{aligned} \tag{5.8}$$

and (u_1,u_2) are the real parts of the components of the coherent states $\boldsymbol{\alpha}$.

6. Correlation coefficient

Before we consider the correlation coefficient we first use the W-Wigner representation to find the rotated variances Δx_ϕ and Δp_ϕ for each mode. To do so we define the rotated coordinates and momenta as

$$x_i=x_{\phi_i}\cos\phi_i-p_{\phi_i}\sin\phi_i,\qquad p_{x_i}=p_{\phi_i}\cos\phi_i+x_{\phi_i}\sin\phi_i,\qquad i=1,2, \tag{6.1}$$

where

$$x_1=\frac{(q_1-q_2)}{\sqrt{2}},\qquad x_2=\frac{(q_1+q_2)}{\sqrt{2}},\qquad p_{x_1}=\frac{(p_1-p_2)}{\sqrt{2}}\qquad\text{and}\qquad p_{x_2}=\frac{(p_1+p_2)}{\sqrt{2}}. \tag{6.2}$$

Therefore, if one uses equation (4.3), equation (6.2) yields

$$\begin{aligned}\Delta x_{\phi_1}&=\left[|f_-|^2\cos^2\phi_1+|k_-|^2\sin^2\phi_1+\tfrac{1}{2}(f_-k_-^*+f^*k_-)\sin 2\phi_1\right]^{-\frac{1}{2}},\\ \Delta p_{\phi_1}&=\left[|f_-|^2\sin^2\phi_1+|k_-|^2\cos^2\phi_1-\tfrac{1}{2}(f_-k_-^*+f^*k_-)\sin 2\phi_1\right]^{-\frac{1}{2}},\end{aligned} \tag{6.3}$$

where

$$\phi_1=\frac{1}{2}\tan^{-1}\left(\frac{(f_-k_-^*+f_-^*k_-)}{(|f_-|^2-|k_-|^2)}\right). \tag{6.4}$$

Similar expressions can be obtained for the second mode provided we replace the sign $(-)$ by $(+)$ in the above equations.

The widths Δx_{ϕ_1} and Δp_{ϕ_1}, equation (6.3) of the Gaussian in the rotated system, reduce to

$$\begin{aligned}\Delta x_{\phi_1}&=\sqrt{2}[(|f_-|^2+|k_-|^2)+\sqrt{(|f_-|^2-|k_-|^2)^2+(f_-k_-^*+f_-^*k_-)^2}]^{-\frac{1}{2}},\\ \Delta p_{\phi_1}&=\sqrt{2}[(|f_-|^2+|k_-|^2)-\sqrt{(|f_-|^2-|k_-|^2)^2+(f_-k_-^*+f_-^*k_-)^2}]^{-\frac{1}{2}}.\end{aligned} \tag{6.5}$$

Moreover one can see from equation (6.5) the phase space area $\Delta x_{\phi_1}\cdot\Delta p_{\phi_1}=1$, which is independent of time.

It is well known that the Wigner function of a Floquet state $|\boldsymbol{n};t\rangle$ rotates in phase space and it becomes squeezed in one direction and elongated in the orthogonal direction. This rotation and squeezing phenomenon leads to a correlation between position and momentum. Indeed the position q_i and the momentum p_i are uncorrelated in the wavefunctions at time

$t = 0$. However, at later times they are coupled in the quadratic form $|z_i|^2$, $i = 1, 2$, of (4.8) via the cross term proportional to $q_i p_i$, or coupled in the quadratic form $|z_\pm|^2$ of (5.3). The two conjugate variables, q_i and p_i, can be expressed in terms of the correlation coefficient [28] as

$$r_i = \frac{\frac{1}{2}\langle(\hat{q}_i\hat{p}_i + \hat{p}_i\hat{q}_i)\rangle - \langle\hat{q}_i\rangle\langle\hat{p}_i\rangle}{\sqrt{\delta q_i \delta p_i}}, \qquad i = 1, 2, \tag{6.6}$$

where

$$\delta q_i = \sqrt{\langle\hat{q}_i^2\rangle - \langle\hat{q}_i\rangle^2}, \qquad \delta p_i = \sqrt{\langle\hat{p}_i^2\rangle - \langle\hat{p}_i\rangle^2}. \tag{6.7}$$

For the first field and after straightfoward calculations the nominator $\digamma_1$ (say) in equation (6.6) is

$$\digamma_1 = \frac{\mathrm{i}}{2} + \frac{1}{4}[k_+(\dot{k}_+^* - \gamma(t)k_+^*) + k_-(\dot{k}_-^* - \gamma(t)k_-^*)], \tag{6.8}$$

while the denominator Υ_1 (say) is

$$\Upsilon_1 = \tfrac{1}{4}[(|k_+|^2 + |k_-|^2)(|\dot{k}_+ - \gamma(t)k_+|^2 + |\dot{k}_- - \gamma(t)k_-|^2)]^{\frac{1}{2}}, \tag{6.9}$$

where we have taken $\hbar = 1$. It is easy to realize that the cross term is dominant in the denominator Υ_1.

From equations (6.8) and (6.9) together with equation (6.6) the correlation coefficient r_1 can be obtained. It should be noted that the expression of the correlation coefficient for the second-mode case is identical to that of the first-mode case. Moreover we emphasise that the correlation coefficient does not depend on the nature of the state which used to calculate the expectation value of the momenta and the coordinates.

Now suppose we consider the case in which the response function, $\gamma(t)$, is a constant as well as the difference between the square of the field frequency, $\omega(t)$, and the coupling parameter, $\lambda(t)$, is also a constant, ω_0^2 (say). In this case we have

$$\text{(i)}\ \frac{\mathrm{d}^2 k_+}{\mathrm{d}t^2} + \left(\omega_0^2 - \gamma^2\right)k_+ = 0 \qquad \text{and} \qquad \text{(ii)}\ \frac{\mathrm{d}^2 k_-}{\mathrm{d}t^2} + \left(\omega_0^2 - \gamma^2 + 2\lambda(t)\right)k_- = 0. \tag{6.10}$$

The first equation is just a simple harmonic motion while the second is the modified Mathieu's equation provided we adjust the coupling parameter to take the form [29]

$$\lambda(t) = \tfrac{1}{2}(\mu\nu^2 \sin\nu t - \mu^2\nu^2\cos^2\nu t), \tag{6.11}$$

where μ and ν are two arbitrary parameters. In order to solve equation (6.10) we first replace k_- by U and substitute (6.11) into (6.10). Thus

$$\frac{\mathrm{d}^2 U}{\mathrm{d}t^2} + \left(\overline{\omega}_0^2 + \mu\nu^2\sin\nu t - \mu^2\nu^2\cos^2\nu t\right)U = 0, \tag{6.12}$$

where $\overline{\omega}_0^2 = \left(\omega_0^2 - \gamma^2\right)$.

By introducing the differential equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}t^2} + \left(\overline{\omega}_0^2 - \mu\nu^2\sin\nu t - \mu^2\nu^2\cos^2\nu t\right)W = 0 \tag{6.13}$$

we are able to split both equations into two coupled first-order differential equations as follows. We have

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \overline{\omega}_0 W + \Gamma(t)U, \qquad \frac{\mathrm{d}W}{\mathrm{d}t} = -\Gamma(t)W - \overline{\omega}_0 U \qquad \text{with} \quad \Gamma(t) = \mu\nu\cos\nu t. \tag{6.14}$$

Now, if we introduce the complex quantity $A = U + \mathrm{i}W$, (6.14) becomes

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\mathrm{i}\overline{\omega}_0 A + \mu\nu\cos\nu t A^* \qquad \text{and} \qquad \frac{\mathrm{d}A^*}{\mathrm{d}t} = \mathrm{i}\overline{\omega}_0 A^* + \mu\nu\cos\nu t A. \tag{6.15}$$

After some manipulation we can write the general solution of equation (6.12) as

$$k_-(t) = C_1 \left(\cos(\nu/2)t \left[\cosh \rho t + \frac{\mu\nu}{2\rho} \sinh \rho t \right] + \frac{\eta}{\rho} \sinh \rho t \sin(\overline{\omega}_0 + \eta/2)t \right)$$
$$+ C_2 \left(\sin(\nu/2)t \left[\cosh \rho t - \frac{\mu\nu}{2\rho} \sinh \rho t \right] - \frac{\eta}{\rho} \sinh \rho t \cos(\overline{\omega}_0 + \eta/2)t \right), \tag{6.16}$$

where we have used the abbreviations

$$\rho = \tfrac{1}{2}\sqrt{\eta^2 - \mu^2\nu^2}, \qquad \eta = (\nu - 2\overline{\omega}_0). \tag{6.17}$$

It should be noted that to obtain the previous result we have applied what is called the rotating wave approximation in which we have neglected the rapidly oscillating term compared with the slowly oscillating term. This is a standard procedure and is justified by the close approximation to zero of a high frequency slowly modulated wave when integrated over an interval comparable to the period of the modulated wave.

7. Conclusion

In this paper we have treated the problem of the construction of the Wigner functions for time-dependent coupled linear oscillators through the use of the linear and quadratic invariants which can be constructed in certain cases of relationships between the time-dependent parameters of the system. Specifically we considered the case of exact resonance in the presence of second harmonic generation. Under these conditions the invariants exist and can be used to construct the Dirac operators which provide the solutions for the wavefunctions in both the coherent and Fock states representations. With the construction of the Wigner states we are able to proceed to the calculation of the correlation functions.

In the calculation of the correlation functions we presented a precise result based on a specific choice of the nature of the time-dependent in the parameters of the system. The ability to do this is not confined to the functions chosen above.

We may introduce another viewpoint for dealing with the time-dependent frequency and time-dependent coupling parameter. For example, if we choose both of them such that

$$\omega^2(t) = \left(\widetilde{\omega}_0^2 - \tfrac{1}{2}\mu^2\nu^2 \cos 2\nu t\right), \qquad \lambda(t) = \mu\nu^2 \sin \nu t, \qquad \widetilde{\omega}_0^2 = \left(\omega_0^2 - \gamma^2 - \tfrac{1}{2}\mu^2\nu^2\right), \tag{7.1}$$

in this case we have that

$$\text{(i)}\ \frac{\mathrm{d}^2 k_+}{\mathrm{d}t^2} + \left(\overline{\omega}_0^2 - \mu\nu^2 \sin \nu t - \mu^2\nu^2 \cos^2 \nu t\right) k_+ = 0, \tag{7.2}$$

and

$$\text{(ii)}\ \frac{\mathrm{d}^2 k_-}{\mathrm{d}t^2} + \left(\overline{\omega}_0^2 + \mu\nu^2 \sin \nu t - \mu^2\nu^2 \cos^2 \nu t\right) k_- = 0. \tag{7.3}$$

The above two equations can each be regarded as a modified Mathieu's equation. They have a general solution when we apply the rotating wave approximation of the form

$$k_-(t) = k_-(0) \left\{ \cos\left(\frac{\nu}{2}t\right) \left[\cosh \rho t + \frac{\mu\nu}{2\rho} \sinh \rho t \right] + \frac{\eta}{\rho} \sinh \rho t \sin\left(\frac{\nu}{2}t\right) \right\}$$
$$+ k_+(0) \left\{ \sin\left(\frac{\nu}{2}t\right) \left[\cosh \rho t - \frac{\mu\nu}{2\rho} \sinh \rho t \right] - \frac{\eta}{\rho} \sinh \rho t \cos\left(\frac{\nu}{2}t\right) \right\} \tag{7.4}$$

and

$$k_+(t) = k_+(0)\left\{\cos\left(\frac{\nu}{2}t\right)\left[\cosh\rho t - \frac{\mu\nu}{2\rho}\sinh\rho t\right] + \frac{\eta}{\rho}\sinh\rho t\sin\left(\frac{\nu}{2}t\right)\right\} - k_-(0)\left\{\sin\left(\frac{\nu}{2}t\right)\left[\cosh\rho t + \frac{\mu\nu}{2\rho}\sinh\rho t\right] - \frac{\eta}{\rho}\sinh\rho t\cos\left(\frac{\nu}{2}t\right)\right\}, \tag{7.5}$$

where ρ is given by equation (6.17). It should be noted that to obtain this and the previous result we have applied the rotating wave approximation, where we have neglected the rapidly oscillating term $\exp[\mathrm{i}(\nu + 2\overline{\omega}_0)]$ compared with the slowly oscillating term $\exp[\mathrm{i}(\nu - 2\overline{\omega}_0)]$.

Our approach here has been to look at the selection of conditions—the exact resonance, for example—based on physical principles which enable further progress of the analysis. We hope to report on a more algorithmic approach in the near future.

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